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On the Convergence of Periodic Splines of Arbitrary Degree

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H. ter Morsche [12] presented a unified theory of interpolation by periodic splines of degree m on a uniform mesh with mesh size $h = 1/n$. He obtained error bounds of the form $\| D^r f - D^r \phi \|_\infty \leqslant K h^{m-r} \| D^m f \|_\infty$ $(0 \leqslant r < m)$ for $f \in C^m(\mathbb{R})$ such that $D^j f(0) = D^j f(1)$ $(0 \leqslant j \leqslant m)$, i.e., $f \in C_0^m$. This extends results of Quade and Collatz [13] and Subbotin [14]. We will establish error estimates of the form

$$\| D^r f - D^r \phi \|_\infty \leqslant K h^{m+1-r} \| D^{m+1} f \|_\infty \qquad (0 \leqslant r \leqslant m)$$

for $f \in C_0^{m+1}$. This generalizes special results of Dubeau and Savoie [5, 6] to arbitrary degree m.

1. BASIC DEFINITIONS AND RESULTS

Let

$$0 = y_0 < y_1 < \cdots < y_{n-1} < y_n = 1 \tag{1.1}$$

be a uniform subdivision of the interval [0, 1], i.e.,

$$y_i = ih, \qquad h = 1/n \qquad (i = 0, ..., n). \tag{1.2}$$

The complex linear space of periodic spline functions of degree m with *spline knots* $y_i = i/n$ $(i \in \mathbb{Z})$ is denoted by $S_0(m, n)$. C_0^k will stand for the class of functions $f \in C^k(\mathbb{R})$ satisfying $D^j f(0) = D^j f(1)$ $(j = 0, ..., k)$. It was shown by ter Morsche that if a spline function $\phi \in S_0(m, n)$ interpolates a given function $f \in C(\mathbb{R})$ with period 1 at the *interpolation points*

$$x_i = y_i - \lambda h \qquad (0 \leqslant \lambda < 1;\ i \in \mathbb{Z}) \tag{1.3}$$

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then it is necessary that the parameters

$$M_i = D^{m-1}\phi(y_i) \tag{1.4}$$

satisfy a linear system of equations called working equations. Denote by Δ the forward difference operator defined by

$$\Delta F(x) = F(x+h) - F(x). \tag{1.5}$$

THEOREM 1.1 [12]. *Let f be a function with period 1. If the linear system*

$$\sum_{r=0}^{m} M_{i-1+r}\left(\sum_{j=0}^{r}(-1)^j\binom{m+1}{j}(r-j+\lambda)^m/m!\right)$$
$$= h^{1-m}\Delta^{m-1}f(x_i) \qquad (i=1, ..., n) \tag{1.6}$$

in the unknowns $M_0, M_1, ..., M_{n-1}$, where $M_{n+k} = M_k$ for all $k \in \mathbb{Z}$, has a unique solution

$$M = (M_0, M_1, ..., M_{n-1})^{\mathrm{T}}, \tag{1.7}$$

then there exists a unique spline $\phi \in S_0(m, n)$ with

$$\phi(x_i) = f(x_i) \qquad (i=1, ..., n). \tag{1.8}$$

Remark. Equations (1.6) are called *working equations.*

We assume $n > m$. If we set

$$d = h^{1-m}(\Delta^{m-1}f(x_1), ..., \Delta^{m-1}f(x_n))^{\mathrm{T}}, \tag{1.9}$$

$$a_r = \sum_{j=0}^{r}(-1)^j\binom{m+1}{j}(r-j+\lambda)^m/m! \qquad (r=0, ..., n-1) \tag{1.10}$$

the working equations can be put into the form

$$AM = d, \tag{1.11}$$

where A is the $n \times n$ circulant matrix $C(a_0, a_1, ..., a_{n-1})$:

$$A = C(a_0, a_1, ..., a_{n-1}). \tag{1.12}$$

Note that

$$a_r = 0 \qquad (r > m). \tag{1.13}$$

Let the polynomial $P_m(z) = P_m(z, \lambda)$ be defined by

$$P_m(z, \lambda) = \sum_{r=0}^{m} a_r z^r. \tag{1.14}$$

Using the special circulant

$$Q = C(0, 1, 0, ..., 0), \tag{1.15}$$

one obtains

$$A = P_m(Q, \lambda) = \sum_{r=0}^{m} a_r Q^r. \tag{1.16}$$

The matrix Q has the n distinct eigenvalues

$$\omega_k = \exp(2\pi ki/n) \qquad (k = 0, ..., n-1). \tag{1.17}$$

Consequently, the eigenvalues of A are given by

$$\lambda_k = P_m(\omega_k, \lambda) \qquad (k = 1, ..., n). \tag{1.18}$$

If none of the eigenvalues λ_k is equal to zero, then the working equations have exactly one solution and spline interpolation is possible in view of Theorem 1.1. This observation leads to the investigation of the generalized Euler–Frobenius polynomial $P_m(z, \lambda)$ [10, 12]. As a consequence the following result holds.

THEOREM 1.2 [12]. *Let there be given a uniform subdivision $y_i = i/n$ $(i \in \mathbb{Z})$ together with a periodic function f with period* 1. *Furthermore, let the interpolation points x_i, $i \in \mathbb{Z}$, be defined as $x_i = y_i - \lambda h$ $(0 \leqslant \lambda < 1)$. Then there exists a uniquely determined periodic spline function $\phi \in S_0(m, n)$ with the interpolation properties*

$$\phi(x_i) = f(x_i) \qquad (i = 1, ..., n) \tag{1.19}$$

in each of the following cases when $P_m(-1, \lambda) \neq 0$:

(i) $n = 2s + 1$ *is odd.*

(ii) $m = 2r + 1$ *is odd and* $\lambda \neq \frac{1}{2}$.

(iii) $m = 2r$ *is even and* $\lambda \neq 0$. (1.20)

Remark. Case (i) and case (ii), $\lambda = 0$, were proved by Quade and Collatz [13]. Case (iii), $\lambda = \frac{1}{2}$, was proved by Subbotin [14]. The complete discussion was carried out by ter Morsche [12].

2. The Convergence Theorem

In order to establish a convergence theorem we need some properties of the matrix A established by ter Morsche [12]. Let B be a $n \times n$ matrix and let the norm of the space of n-tuples $x = (x_1, ..., x_n)$ be the sup-norm, i.e.,

$$|x| = \sup_{1 \leqslant i \leqslant n} |x_i|. \tag{2.1}$$

Then the induced norm of the matrix B is the row-maximum-norm:

$$|B| = \sup_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |b_{ij}|. \tag{2.2}$$

Let the numbers λ and m be given such that

$$P_m(-1, \lambda) \neq 0. \tag{2.3}$$

Then $A = P_m(Q, \lambda)$ has an inverse A^{-1} and interpolation in $S_0(m, n)$ at the points $x_i = (i - \lambda)/n$ $(i \in \mathbb{Z})$ is possible. It was shown by ter Morsche [12] that the polynomial $P_m(z, \lambda)$ has m distinct nonpositive roots:

$$\begin{aligned} P_m(z, \lambda) &= a_m(z + z_1)(z + z_2) \cdots (z + z_m), \\ 0 = z_1 < z_2 < &\cdots < z_j < 1 < z_{j+1} < \cdots < z_m. \end{aligned} \tag{2.4}$$

Using the Laurent series of $1/P_m(z, \lambda)$ in $|z_j| < |z| < |z_{j+1}|$ ter Morsche [12] established the estimate:

$$|A^{-1}| \leqslant 1/|P_m(-1, \lambda)|. \tag{2.5}$$

It is of basic importance that the right-hand side of (2.5) is independent of n. Moreover, we need the following properties of $P_m(z, \lambda)$ [12]:

$$\sum_{r=0}^{n-1} a_r = 1, \qquad a_r \geqslant 0 \qquad (r = 0, ..., n-1). \tag{2.6}$$

THEOREM 2.1. *Let the numbers λ and m be given such that*

$$P_m(-1, \lambda) \neq 0. \tag{2.7}$$

Assume that $f \in C_0^{m+1}$ and let $\phi \in S_0(m, n)$ be the interpolating spline function satisfying

$$\phi(x_i) = f(x_i) \qquad (i = 1, ..., n), \tag{2.8}$$

where $x_i = (i - \lambda)/n$. Then we have the error estimates

$$\|D^r\phi - D^r f\|_\infty \leqslant K\left(\frac{1}{n}\right)^{m+1-r} \|D^{m+1} f\|_\infty \qquad (r = 0, ..., m), \tag{2.9}$$

where the constant K depends on m and λ.

Proof. We recall the working equations

$$\sum_{r=0}^{n-1} a_r M_{i-1+r} = d_i \qquad (i = 1, ..., n) \tag{2.10}$$

with

$$M_j = D^{m-1}\phi(y_j), \qquad d_i = h^{1-m}\Delta^{m-1} f(x_i). \tag{2.11}$$

Since f and ϕ have period 1 the relations (2.10) and (2.11) hold for all integers i, j. We set

$$N_j = (M_{j+1} - M_j)/h, \qquad e_i = (d_{i+1} - d_i)/h \tag{2.12}$$

and obtain from (2.10) by subtracting members of each equation from corresponding members of its successor the equations

$$\sum_{r=0}^{n-1} a_r N_{i-1+r} = e_i \qquad (i = 1, ..., n). \tag{2.13}$$

Thus, if $A = P_m(Q, \lambda) = C(a_0, a_1, ..., a_{n-1})$ and

$$N = (N_0, N_1, ..., N_{n-1})^{\mathrm{T}}, \qquad e = (e_1, ..., e_n)^{\mathrm{T}}, \tag{2.14}$$

we have

$$A(N - e) = (I - A)\, e =: F \tag{2.15}$$

with

$$F = (F_1, ..., F_n)^{\mathrm{T}}. \tag{2.16}$$

Since

$$e_{i+n} = e_i \qquad (i \in \mathbb{Z}) \tag{2.17}$$

we obtain

$$F_i = e_i - \sum_{r=0}^{n-1} a_r e_{i+r} \qquad (i = 1, ..., n). \tag{2.18}$$

Taking into account (2.6) and (1.13) we get

$$F_i = \sum_{r=1}^{m} a_r(e_i - e_{i+r}) \qquad (i = 1, \ldots, n). \tag{2.19}$$

Since

$$e_i = (\Delta^{m-1}f(x_{i+1}) - \Delta^{m-1}f(x_i))/h^m = \Delta^m f(x_i)/h^m \tag{2.20}$$

there are real numbers u_i satisfying

$$e_i = D^m f(u_i), \qquad x_i \leqslant u_i \leqslant x_{i+m} \qquad (i = 1, \ldots, n). \tag{2.21}$$

Then we have

$$\begin{aligned} |e_i - e_{i+r}| &= |D^m f(u_i) - D^m f(u_{i+r})| \\ &\leqslant \omega\left(D^m f, \frac{1}{n} 2m\right), \end{aligned}$$

i.e.,

$$|e_i - e_{i+r}| \leqslant 2m\omega\left(D^m f, \frac{1}{n}\right) \qquad (i = 1, \ldots, n;\ r = 1, \ldots, m), \tag{2.22}$$

where $\omega(D^m f, 1/n)$ denotes the modulus of continuity of $D^m f$. Now it follows from (2.22), (2.19), and (2.6) that

$$|F| \leqslant 2m\omega\left(D^m f, \frac{1}{n}\right). \tag{2.23}$$

Since $A(N - e) = F$ we obtain from (2.5) and (2.23)

$$|N - e| \leqslant \omega\left(D^m f, \frac{1}{n}\right) 2m / |P_m(-1, \lambda)|. \tag{2.24}$$

For $x_{i-1} < x < x_i$ we have

$$|D^m \phi(x) - e_i| = |N_{i-1} - e_i| \tag{2.25}$$

in view of $D^m \phi(x) = (M_i - M_{i-1})/h$. It follows from (2.21) that

$$|D^m f(x) - e_i| = |D^m f(x) - D^m f(u_i)|$$

which implies

$$|D^m f(x) - e_i| \leqslant (m+1)\,\omega\left(D^m f, \frac{1}{n}\right) \qquad (x_{i-1} \leqslant x \leqslant x_i). \tag{2.26}$$

Taking into account (2.24), (2.25), and (2.26) we can conclude for $x_{i-1} < x < x_i$,

$$|D^m\phi(x) - D^m f(x)| \leqslant |D^m\phi(x) - e_i| + |D^m f(x) - e_i|$$
$$\leqslant \omega\left(D^m f, \frac{1}{n}\right)((m+1) + 2m/|P_m(-1, \lambda)|)$$

which implies

$$\|D^m\phi - D^m f\|_\infty \leqslant K\omega\left(D^m f, \frac{1}{n}\right), \tag{2.27}$$

where the constant K depends only on m and λ. For $f \in C_0^{m+1}$ we obtain from (2.27)

$$\|D^m\phi - D^m f\|_\infty \leqslant \frac{K}{n} \|D^{m+1} f\|_\infty. \tag{2.28}$$

This establishes the estimate (2.9) for $r = m$. By integration and the interpolation property of ϕ the estimates for smaller values of r may be established in the usual manner.

The error estimates of Theorem 2.1 were proved by Ahlberg, Nilson, and Walsh [1] for the case $m = 2r + 1$, $\lambda = 0$ (see also Golomb [7] who investigated also the mean square error). The cases $m = 2, 4$; $\lambda = 0$; $n = 2s + 1$ were proved by Dubeau and Savoie [5, 6]. Theorem 2.1 shows that these special cases extend to all periodic interpolating splines on uniform meshes which include in particular Subbotin's midpoint splines of even degree.

3. Construction of the Interpolant

Let $B_q(t)$ denote the Bernoulli function of degree q. Its Fourier series is given by

$$B_q(t) = \sum_{|r|>0} (2\pi i r)^{-q} \exp(2\pi i r t). \tag{3.1}$$

$B_q(t)$ is a polynomial of degree q in $0 < t < 1$ which can be constructed by the recursion

$$B_0(t) = 1, \qquad B_q'(t) = B_{q-1}(t), \qquad \int_0^1 B_q(t)\,dt = 0. \tag{3.2}$$

Using a result of Wegner [15] we have shown in [4] that the function M defined by

$$M(t) = 1 - h^{-m-1} \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^{m+1-j} B_{m+1}(y_j - t) \qquad (3.3)$$

is the 1-periodic B-spline of degree m with knots $y_0, ..., y_{m+1}$. It is well known [9] that the spline space $S_0(m, n)$ $(n > m)$ is spanned by the translates of M:

$$S_0(m, n) = \text{lin}\{M(\cdot - y_j): j = 0, ..., n-1\}. \qquad (3.4)$$

We will show that a modification of the method of interpolation by translation [2, 3, 8] yields a construction of the spline interpolant ϕ of f. It follows from (3.4) and Theorem 1.2 that there exist unique $a_0, ..., a_{n-1}$ such that

$$\phi(t) = \sum_{j=0}^{n-1} a_j M(t - y_j) \qquad (3.5)$$

and

$$\phi(x_k) = f(x_k) \qquad (k = 1, ..., n). \qquad (3.6)$$

Recall that

$$x_k = y_k - \lambda h \qquad (h = 1/n;\ 0 \leqslant \lambda < 1). \qquad (3.7)$$

We introduce the functions

$$\psi(t) = \phi(t - \lambda h), \qquad g(t) = f(t - \lambda h),$$
$$N(t) = M(t - \lambda h). \qquad (3.8)$$

Then the shifted spline ψ is the unique function in

$$\text{lin}\{N(\cdot - y_j): j = 0, ..., n-1\} \qquad (3.9)$$

such that

$$\psi(y_k) = g(y_k) \qquad (k = 0, ..., n-1). \qquad (3.10)$$

Thus, the method of interpolation by translation is applicable. We list the corresponding formulas:

$$b_k = \sum_{j=0}^{n-1} N(-y_j) \exp(2\pi i k y_j),$$

$$c_k = (1/n) \sum_{j=0}^{n-1} \exp(2\pi i j y_k)/b_j, \qquad a_k = \sum_{j=0}^{n-1} c_j g(y_{k-j}). \qquad (3.11)$$

As an example we consider the function

$$f(t) = \sin(\sin(2\pi t)).$$

We computed the discrete maximum norm of the error function $f - \phi$ at the mesh $j/30$ ($j = 1, ..., 29$):

$1.11445E-02$	4
$1.84359E-02$	8
$4.30828E-03$	12
$1.08194E-03$	16
$7.94917E-04$	20

$\|f-\phi\|_\infty$ for $\phi \in S_0(2, n)$ ($n = 4, 8, 12, 16, 20$; $\lambda = 0.25$);

$9.18536E-02$	4
$1.00384E-02$	8
$1.10123E-03$	12
$2.00838E-04$	16.

$\|f-\phi\|_\infty$ for $\phi \in S_0(3, n)$ ($n = 4, 8, 12, 16$; $\lambda = 0.25$).

Remark. An alternative construction of periodic interpolating splines of arbitrary degree is given by Merz [11].

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